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Killing spinors, twistor - spinors and Hijazi inequality

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Dedicated to I.M. Gelfand on his 75th birthday

Abstract. *Let (W, g) be a spin manifold of dimension n. In terms of the Dirac operator P of (W, g), we introduce on the spinor fields a conformally covariant first-order operator D that is strictly connected with the twistor-spinors. We show that the operator $(\Delta - \rho)$ $(\rho = (n/4(n-1))R)$ is positive. For a compact spin manifold of dimension $n \geqslant 3$, the existences of harmonic spinors and twistor-spinors $\not\equiv 0$ are mutually exclusive, except for the parallel spinors. By means of a universal formula, we show that the Hijazi inequality* [8] *holds for every spinor field such that $(P\psi, P\psi) = \lambda^2(\psi, \psi)$ $(\lambda = const)$. In the limiting case, the manifold admits a Killing spinor wich can be evaluated in terms of ψ. Using the Yamabe-Schoen theorem* [15], *we prove that, if the space $\mathscr{K}$ of the twistor-spinors of (W, g) is not reduced to zero, there is a conformal change of the metric g giving a manifold with Killing spinors $\neq 0$. Interpretation of $\dim \mathscr{K}$ in terms of these spaces of Killing spinors. If the compact spin manifold (W, g) of dimension $n \geqslant 3$ is not conformally isometric with the sphere, every twistor-spinor is without zero on W.*

INTRODUCTION

Killing spinors were first introduced in Mathematical Physics: general relativity, 11-dimensional (resp. 10-dimensional) supergravity theory, supersymmetry, matter fields (see, for example [1] to [5]). The context is often the following:

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the main space is a fiber bundle on a space-time; the fibers are properly Riemannian manifolds (for example spheres or homogeneous spaces) admitting a spinor structure. In many cases, *Killing spinors* appear on these manifolds.

The notion also appears in a purely geometrical way in direct relation with the Dirac operator P of a spin manifold. A Killing spinor is automatically an eigenspinor of P and is a generalization of the notion of parallel spinor. Many years ago, I studied properties of parallel spinors in the context of the *harmonic spinors* [6]. Some interesting steps in this direction has been recently taken by Friedrich [7] and Hijazi [8].

The notion of Killing spinor is a particular case of the notion of *twistor-spinor* which has been introduced by Penrose [10]. I have defined recently [12] a *conformally covariant* first-order operator D on the spinors such that the twistor-spinors are the zeros of D. This operator appears in an universal formula that is the main tool of this paper (formula (2-5)).

For a *compact* spin manifold, Hijazi has given a lower bound for the square of the eigenvalues of P in terms of conformal geometry. The limiting case is precisely the case when the manifold admits Killing spinors. I shall place here the Hijazi inequality in its true context, showing that, in a suitable sense, this inequality is universal. My approach differs from that a Hijazi (see Theorem 4 and Theorem 5). I will show that, roughly speaking, harmonic spinors and twistor-spinors have a character mutually exclusive.

It is possible to interpret the dimension of the space of the twistor-spinors in terms of the dimensions of the spaces of Killing spinors corresponding to a suitable conformal metric given by the Yamabe-Schoen theorem [15]. We prove with the same tool that if a manifold is not conformally isometric with the sphere, every twistor-spinor is without zero.

According to the Hijazi inequality, the physical philosophy which appears can be given in the following way; consider a compact spin manifold as a basis for a Euclidean model for fermionic field ψ. If the manifold admits Killing spinors, we can consider the corresponding eigenvalue ν_1 of the Dirac operator P as associate with a ground state: if ψ satisfies $(P\psi, P\psi) = \lambda^2(\psi, \psi)$, we have $\lambda^2 \geqslant \nu_1^2$.

1. DEFINITIONS AND GENERAL FORMULAS

1. Spin manifolds and corresponding connections

a) Let (W, g) be an oriented Riemannian manifold of dimension $n \geqslant 2$ (with, $n = 2\nu$ or $n = 2\nu + 1$) endowed with a positive definite metric g. If (W, g) has a *spin structure* with *fundamental vector-spinor* γ, there is a principal Spin(n)-bundle $\mathscr{E}$ on W which is a two-fold covering $p : \mathscr{E} \to E$ of the orthonormal oriented frames bundle of (W, g) of structural group $SO(n)$; a point z is called a spin frame. The manifold (W, g) being referred to the orthonormal frames, introduce the $2^\nu \times 2^\nu$ Dirac matrices satisfying:

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = -2\, g_{\alpha\beta}\, e \qquad (\alpha, \beta = 1, \ldots, n). \tag{1.1}$$

These matrices give the components of γ with respect to the spin frames. It is well known that the matrices γ_α can be chosen *antihermitian* ($\widetilde{\gamma}_\alpha = -\gamma_\alpha$, where $\sim$ denotes the adjunction). The group Spin(n) can be considered as a group of $2^\nu \times 2^\nu$ *unitary matrices* Λ (with $\widetilde{\Lambda} = \Lambda^{-1}$) satisfying:

$$\Lambda \gamma_\alpha \Lambda^{-1} = A_\alpha^{\lambda'}\, \gamma_{\lambda'} \tag{1.2}$$

where $A = (A_\alpha^\lambda)$ is the projection of Λ on $SO(n)$.

If ψ is a contravariant 1-spinor ($\psi(z\, \Lambda^{-1}) = \Lambda\, \psi(z)$, where z is a spinor frame), $\widetilde{\psi}$ is a covariant 1-spinor ($\widetilde{\psi}(z\, \Lambda) = \widetilde{\psi}(z)\, \Lambda$) and the space Σ_x ($x \in W$) of the contravariant 1-spinors at x admits a canonical structure of Hermitian vector space given by the scalar product $(\psi^{(1)}, \psi^{(2)}) = \widetilde{\psi}^{(2)}, \psi^{(1)}$. We denote by Σ the vector bundle of the contravariant 1-spinors of (W, g).

b) The Riemannian connection of (W, g) induces a connection on the principal bundle $\mathscr{E}$ by the isomorphism between the Lie algebras of Spin(n) and $SO(n)$. Now introduce local section of the orthonormal frames bundle. If $\omega_U = (\omega_{\alpha\beta})$ (with $\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0$) is, for a domain U of W, the 1-form defining the Riemannian connection, the corresponding spinor connection is given by the 1-form

$$\sigma_U = -(1/4)\, \omega_{\alpha\beta}\, \gamma^\alpha \gamma^\beta \tag{1.3}$$

We denote by ∇ the corresponding covariant differentiation. We have $\nabla \gamma = 0$ and the adjunction commutes with ∇. Let $RC = (R_{\alpha\beta,\ \lambda\mu})$ be the Riemannian curvature tensor of (W, g), $Ri = (R_{\alpha\beta})$ its Ricci tensor and R its scalar curvature. We have on U the classical formulas:

$$R_{\alpha\beta,\lambda\mu}\, \gamma^\beta \gamma^\lambda \gamma^\mu = 2\, R_{\alpha\beta} \gamma^\beta \qquad R_{\alpha\beta} \gamma^\alpha \gamma^\beta = -R.$$

For a spinor field ψ, we have the Ricci identity:

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\, \psi = -(1/4)\, R_{\lambda\mu,\alpha\beta}\, \gamma^\lambda \gamma^\mu\, \psi$$

It follows that:

$$\gamma^\beta (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\, \psi = -(1/2)\, R_{\alpha\beta}\, \gamma^\beta\, \psi \tag{1.4}$$

c) Let P be the *Dirac operator* on (W, g), $\widetilde{P}$ its adjoint. We have

$$P\psi = \gamma^\alpha \nabla_\alpha \psi \qquad \widetilde{P}\widetilde{\psi} = -\nabla_\alpha \widetilde{\psi} \gamma^\alpha. \tag{1.5}$$

We set $\Delta\psi = P^2\ \psi$ (Laplacian of spinor). An elementary calculus gives [1]

(1.6) $$\Delta\psi = -\nabla^\alpha\nabla_\alpha\psi + (R/4)\,\psi.$$

If W is *compact*, introduce the global scalar product $\langle\psi^{(1)}, \psi^{(2)}\rangle = \int_W (\psi^{(1)}, \psi^{(2)})\ \eta$ where η is the Riemannian volume element. We have similar scalar products on the tensor-spinors; P is formally selfadjoint with respect to this scalar product and we have:

$$\langle\Delta\psi^{(1)}, \psi^{(2)}\rangle = \langle P\,\psi^{(1)}, P\,\psi^{(2)}\rangle = \langle\psi^{(1)}, \Delta\,\psi^{(2)}\rangle$$

We seee that *the spectrum of P is real* and that Δ is *positive* selfadjoint.

d) If there is on (W, g) a spinor field ψ without zero, set

(1.7) $$(P\psi,\ \ P\psi) = r^2(\psi, \psi) \qquad (r \geqslant 0)$$

If r is > 0, the spinor $r^{-1}P\psi$ admits same square as ψ. Suppose that there exists $\Lambda : x \in W \to \Lambda(x) \in \mathrm{Spin}(n)$ such that

(1.8) $$P\psi = r\,\Lambda\,\psi$$

We will study spinor fields (having or not zeroes) satisfying (1.7) or (1.8), r being $\geqslant 0$. If ψ satisfies (1.8), introduce a corresponding connection on the vector bundle Σ given by $\sigma_U^{(r,\Lambda)} = \sigma_U + (r/n)\,\gamma\,\Lambda\,|_U$. We denote by $\nabla^{(r,\Lambda)}$ the corresponding covariant differentiation. We have:

$$\nabla^{(r,\Lambda)}\psi = \nabla\,\psi + (r/n)\,\gamma\,\Lambda\,\psi$$

A field ψ such that $\nabla^{(r,\Lambda)}\psi = 0$ satisfies (1.8). We now have (see [9]).

PROPOSITION 1. *Any field $\psi \not\equiv 0$ such that $\nabla^{(r,\Lambda)}\ \psi = 0$ has no zero on W.*

In fact, consider a domain U of W with a local section $\{e_\alpha\}$ of E and denote by $\{\theta^\lambda\}$ the dual of $\{e_\alpha\}$; on U we set $\omega_{\alpha\beta} = \omega_{\alpha\beta,\lambda}\theta^\lambda$. Suppose that ψ admits a zero on W; we can find on such a domain U two points x_1, x_2 connected by a path $m(t)$ such that $\psi(x_1) = 0$ and $\psi(x_2) \neq 0$. If $m(t) = M^\alpha(t)\,e_\alpha$, $\psi \circ m$ satisfy the linear differential equation with smooth coefficients

$$(d/dt)\,(\psi \circ m) - (1/4)\,M^\lambda(t)\,\omega_{\alpha\beta,\lambda}\gamma^\alpha\gamma^\beta(\psi \circ m) +$$
$$+ (r/n)\,M^\lambda(t)\,\gamma_\lambda\ \Lambda(\psi \circ m) = 0$$

We deduce from the nullity of the initial condition at x_1 that $\psi \circ m \equiv 0$ and ψ cannot be $\not\equiv 0$. ∎

e) Take $r = \text{const.}$, $\Lambda = e^{i\theta}\,Id$ where $\theta = \text{const.}$ and set $\nu_1 = r\,e^{i\theta} \in \mathbf{C}$.

A Killing spinor of (W, g) is a spinor $\psi \not\equiv 0$ such that

(1.9) $$\nabla^{(\nu_1)}\psi = \nabla\psi + (\nu_1/n)\,\gamma\psi = 0$$

where ν_1 is a complex constant. We have then $P\psi = \nu_1\,\psi$ and ν_1 is an eigenvalue of P. If $\nu_1 = 0$ we have a parallel spinor field. We say that a Killing spinor is *non trivial* if $\nu_1 \neq 0$.

2. The operator D defining the twistor-spinors and the universal formula

a) We introduce systematically in the following the first order differential operator D on the spinors defined by:

(2.1) $$D\psi = \nabla\psi + (1/n)\,\gamma\,P\psi$$

which satisfies $\gamma^\alpha D_\alpha\psi = 0$. If ψ satisfies (1.8) $\nabla^{(r,\Lambda)}\psi = D\psi$. It is well-known that the twistor equation ([10] up to notations) can be written

(2.2) $$\gamma_\alpha\nabla_\beta\,\psi + \gamma_\beta\nabla_\alpha\psi = (2/n)\,g_{\alpha\beta}\,P\psi$$

By multiplication by γ^β, it follows from (2.2) that:

$$-(n+2)\,\nabla_\alpha\psi - \gamma_\alpha P\psi = (2/n)\,\gamma_\alpha\,P\psi$$

that is:

$$(n+2)\,(\nabla_\alpha\psi + (1/n)\,\gamma_\alpha\,P\psi) = 0$$

let $D\psi = 0$. The converse is clear. Therefore $D\psi = 0$ says that ψ satisfies the twistor equation. *A spinor ψ satisfying $D\psi = 0$ is called here a twistor-spinor.*

b) Let ψ be an arbitrary spinor field. We associate with ψ the real scalar and the real 1-form given by

$$u(\psi) = u = (\psi, \psi) = \tilde{\psi}\,\psi \geqslant 0 \quad T(\psi) = \tilde{\psi}\,\gamma\,P\psi - \tilde{P}\,\tilde{\psi}\,\gamma\,\psi$$

Let us evaluate the positive Laplacian of u:

$$\Delta u = -\nabla^\alpha\nabla_\alpha(\tilde{\psi}\,\psi) = -(\nabla^\alpha\nabla_\alpha\tilde{\psi})\,\psi - \tilde{\psi}(\nabla^\alpha\nabla_\alpha\psi) - 2\nabla^\alpha\tilde{\psi}\nabla_\alpha\psi$$

let:

(2.3) $$\Delta u = (\Delta\tilde{\psi})\,\psi + \tilde{\psi}(\Delta\psi) - (R/2)u - 2(\nabla\psi, \nabla\psi)$$

But we have

(2.4) $$\tilde{\psi}(\Delta\psi) = \tilde{\psi}\gamma^\alpha\nabla_\alpha P\psi = \nabla_\alpha(\tilde{\psi}\gamma^\alpha\,P\psi) + (P\psi, P\psi)$$

It follows from (2.3), (2.4) that the codifferential of the 1-form $T(\psi)$ satisfies

$$(1/2)\,\Delta u + (1/2)\,\delta\, T(\psi) = -\,(R/4)u + (P\psi, P\psi) - (\nabla\psi, \nabla\psi)$$

We deduce from (2.1):

$$(\nabla\psi, \nabla\psi) = (D\psi, D\psi) + (1/n)\,(P\psi, P\psi)$$

We obtain thus *the universal formula*, valid for an arbitrary field ψ:

(2.5) $$(1/2)\Delta u + (1/2)\,\delta\, T(\psi) = -\,(R/4)u + ((n-1/n)(P\psi, P\psi)\text{-}(D\psi, D\psi)$$

c) By differentiation of (2.1), we have:

$$-\,\nabla^\alpha D_\alpha\,\psi = -\,\nabla^\alpha\nabla_\alpha\,\psi - (1/n)\,\Delta\,\psi$$

that is:

(2.6) $$-\,\nabla^\alpha D_\alpha\,\psi = ((n-1)/n)(\Delta\psi - \rho\psi) \qquad (\rho = (n/4(n-1))R)$$

Suppose W is *compact*. We obtain by integration of (2.5) on (W, g)

$$((n-1)/n) \quad \langle P\psi, P\psi\rangle = \langle (R/4)\,\psi, \psi\rangle + \langle D\psi, D\psi\rangle$$

that is

(2.7) $$\langle \Delta\psi - \rho\psi, \psi\rangle = (n/(n-1))\,\langle D\psi, D\psi\rangle$$

Therefore it follows from (2.6), (2.7)

PROPOSITION 2. *On a spin manifold, every twistor-spinor ψ (with $D\psi = 0$) satisfies*

(2.8) $$\Delta\psi - \rho\psi = 0 \qquad (\rho = (n/4\,(n-1))R)$$

If W is compact, the operator $(\Delta - \rho)$ on the spinors is a positive operator. Every solution of (2.8) is a twistor-spinor. ■

In the following part of this paragraph, we consider only *twistor-spinors*. If $D\psi = 0$, an elementary computation gives:

(2.9) $$n\,du = -\,T(\psi) \qquad \delta\, T(\psi) = -\,n\,\Delta u$$

It follows from (2.5) that:

(2.10) $$(n/2)\,\Delta u = \rho u - (P\psi, P\psi)$$

From (1.4) and $D\psi = 0$, we deduce:

$$((n-2)/n)\,\nabla_\alpha\, P\psi - (1/n)\,\gamma_\alpha\,\Delta\psi + (1/2)\,R_{\alpha\beta}\gamma^\beta\,\psi = 0$$

and it follows from (2.8) that

(2.11) $$((n-2)/n)\,\nabla_\alpha P\psi - (\rho/n)\,\gamma_\alpha \psi + (1/2)\,R_{\alpha\beta}\,\gamma^\beta \psi = 0$$

Introduce the real 1-form $J_\alpha(\psi) = i\ \tilde{\psi}\gamma_\alpha\psi$. Taking the product of (2.11) by ψ, we obtain

$$(n-2)\nabla_\alpha(\tilde{\psi}\,P\psi) + ((n-2)/n) i\,J_\alpha(P\psi) + i\,\rho\,J_\alpha(\psi) - i\,(n/2)\,R_{\alpha\beta}J^\beta(\psi) = 0$$

It follows by adjunction and addition that:

$$(n-2)\,\nabla_\alpha\,(\tilde{\psi}(P\psi) + (\tilde{P}\tilde{\psi})\psi) = 0$$

We have:

PROPOSITION 3. *Let (W, g) be a spin manifold of dimension $n \geqslant 3$. Every twistor-spinor ψ satisfies:*

(2.12) $$\tilde{\psi}(P\psi) + (\tilde{P}\tilde{\psi})\psi = \text{const.}$$ ∎

3. Conformal change of metric

a) Introduce on W a conformal metric $\bar{g} = \exp(2c)\,g$, where c is a real-valued function. The quantities corresponding to $(W, \bar{g})$ are covered by a bar. A spin structure γ of (W, g) corresponds to a spin structure γ of $(W, \bar{g})$ and a spinor ψ of (W, g) to a spinor $\bar{\psi}$ of $(W, \bar{g})$; in this correspondance the vector-spinor $\bar{\gamma}$, referred to orthonormal frames of (W, g) has components given by $\exp(-\,c)\ \gamma^\alpha$. It is known (see [11] that for any spinor field ψ we have):

(3.1) $$\bar{\nabla}_\alpha \bar{\psi} = \overline{\nabla_\alpha \psi} - (1/2)\,\partial_\beta c\,\overline{\gamma_\alpha \gamma^\beta \psi} - (1/2)\,\partial_\alpha c\,\bar{\psi}$$

If follows that:

(3.2) $$\bar{P}\bar{\psi} = \exp(-c)\,[\overline{P\psi} + ((n-1)/2)\,\partial_\beta c\,\overline{\gamma^\beta \psi}]$$

and thus

(3.3) $$\bar{P}\left[\exp\left(-\,\frac{n-1}{2}\,c\right)\bar{\psi}\right] = \exp\left(-\,\frac{n+1}{2}\,c\right)\overline{P\psi}$$

We deduce from (3.1), (3.2), that:

$$\bar{D}_\alpha \bar{\psi} = \overline{D_\alpha \psi} - (1/2n)\,\partial_\beta c\,\overline{\gamma_\alpha \gamma^\beta{}_r\,\psi} - (1/2)\,\partial_\alpha c\ \psi$$

and therefore:

(3.4) $$\bar{D}\left[\exp\left(\frac{c}{2}\right)\bar{\psi}\right] = \exp\left(\frac{c}{2}\right)\overline{D\psi}$$

Let $\mathcal{K}$ be the space of the twistor-spinor of (W, g). If W is *compact*, $\mathcal{K}$ is the space of the solutions of the elliptic equation (2.8) and the complex dimension of $\mathcal{K}$ is finite. We obtain

THEOREM 1. *The operator D is conformally covariant. In particular if W is compact, the complex dimension of the space $\mathcal{K}$ of the twistor-spinors is a conformal invariant of (W, g).* ∎

2. KILLING SPINORS

4. Properties of spinors ψ such that $\nabla^{(f)}\psi = 0$

a) Let f be a complex-valued function on W; we set $f = re^{i\theta} = a + ib$ where r is $\geqslant 0$ and where a, b are real-valued functions. For $\Lambda = e^{i\theta}\, Id$ the covariant derivative $\nabla^{(r,\Lambda)}\psi$ can be written

$$(4.1) \qquad \nabla^{(f)}\psi = \nabla\psi + (f/n)\,\gamma\psi$$

Study the spin manifolds of dimension $n \geqslant 2$ which admit spinors $\psi \not\equiv 0$ such that $\nabla^{(f)}\psi = 0$, that implies $P\psi = f\psi$; the twistor-spinor ψ is without zero and $u(\psi)$ is > 0. It follows from (1.4) that we have for a twistor-spinor:

$$\nabla_\alpha P\psi + (1/n)\,\nabla_\beta\ (\gamma^\beta\gamma_\alpha P\psi) + (1/2)\,R_{\alpha\beta}\gamma^\beta\psi = 0$$

which can be written for $\nabla^{(f)}\psi = 0$

$$(4.2) \quad \nabla_\alpha f\cdot\psi + (1/n)\nabla_\beta f\cdot\gamma^\beta\gamma_\alpha\psi + (1/2)\,(R_{\alpha\beta} - (4(n-1)/n^2)f^2 g_{\alpha\beta})\,\gamma^\beta\psi = 0$$

Multiplying by γ^α, we get:

$$(4.3) \qquad \nabla_\alpha f\gamma^\alpha\psi - (\rho - f^2)\,\psi = 0$$

b) From (4.1) we deduce that if ψ satisfies $\nabla^{(f)}\psi = 0$, we have:

$$(4.4) \qquad n\,du(\psi) = -\,2b\,J(\psi), \qquad \nabla_\alpha J_\beta(\psi) + \nabla_\beta J_\alpha(\psi) = -\,(4b/n)\,u\,g_{\alpha\beta}$$

Now multiplying (4.3) by $\tilde{\psi}$ and separating real and purely imaginary parts, we get:

$$(4.5) \qquad J^\alpha(\psi)\,\nabla_\alpha a - 2\,abu = 0$$

and

$$(4.6) \qquad J^\alpha(\psi)\,\nabla_\alpha b - (\rho - a^2 + b^2)\,u = 0$$

If $n \geqslant 3$, (2.12) takes the simple form:

$$(4.7) \qquad au = C = \text{const.}$$

5. Killing-spinors

a) Let $\psi \not\equiv 0$ be a Killing spinor of (W, g) of dimension $n \geqslant 2$. We can take $f = \nu_1 \in \mathbb{C}$. It follows from (4.5) that ν_1 is either real or purely imaginary and we deduce from (4.2), (4.3) that:

$$(5.1) \qquad R_{\alpha\beta} = (R/n)\, g_{\alpha\beta} \qquad \nu_1^2 = \rho = (n/4(n-1))\, R$$

In particular, if ψ is a parallel spinor field ($\nu_1 = 0$), the Ricci tensor of (W, g) is null. We consider here non trivial Killing spinors ($\nu_1 \neq 0$).

If W is compact, ν_1 is necessarily real since the spectrum of P is real. Conversely if $\nu_1 \neq 0$ is real, the Ricci tensor is positive definite and W is compact according to a classical theorem of Myers.

We have the following well-known proposition [9]

PROPOSITION 4. *If (W, g) of dimension $n \geqslant 2$ admits a non zero Killing spinor, it is an Einstein space and $\nu_1^2 = \rho$. If $\nu_1 \neq 0$ is real, W is compact; if $\nu_1 \neq 0$ is purely imaginary, W is noncompact.* ■

If ν_1 is real, it follows from (4.4) that $u(\psi) = \text{const.}$ and if $J(\psi) \not\equiv 0$, it defines an infinitesimal isometry, that is a Killing vector. It is the origin of the name of Killing spinor.

b) Consider a spinor $\psi \not\equiv 0$ such that $\nabla^{(f)}\psi = 0$. Multiplying (4.3) by γ_β, we have:

$$\nabla_\beta f \cdot \psi + (\rho - f^2)\,\gamma_\beta \psi + \frac{1}{2}\,\nabla_\alpha f\,(\gamma^\alpha\gamma_\beta - \gamma_\beta\gamma^\alpha)\,\psi = 0$$

and multiplying by $\tilde{\psi}$, we obtain

$$(5.2) \qquad u\nabla_\beta f - i(\rho - f^2) J_\beta(\psi) + \frac{1}{2}\,\nabla_\alpha f \cdot \tilde{\psi}(\gamma^\alpha\gamma_\beta - \gamma_\beta\gamma^\alpha)\psi = 0$$

where the factor $\tilde{\psi}(\gamma^\alpha\gamma_\beta - \gamma_\beta\gamma^\alpha)\psi$ is purely imaginary. Separating the real and purely imaginary parts of (5.2), we have:

$$(5.3) \qquad u\,\nabla_\beta a - 2\,ab\,J_\beta(\psi) + \frac{i}{2}\,\nabla_\alpha b\;\tilde{\psi}(\gamma^\alpha\gamma_\beta - \gamma_\beta\gamma^\alpha)\psi = 0$$

and

$$(5.4) \qquad u\,\nabla_\beta b - (\rho - a^2 + b^2)\,J_\beta(\psi) - \frac{i}{2}\,\nabla_\alpha a\tilde{\psi}(\gamma^\alpha\gamma_\beta - \gamma_\beta\gamma^\alpha)\psi = 0$$

Suppose that f is a real-valued function ($f = a$). It follows from (5.3) that $a =$ const. Therefore ψ is a Killing spinor corresponding to a real ν_1. We have obtained by another method:

PROPOSITION 5. *(Hijazi [8]). On the spin manifold (W, g) of dimension $n \geqslant 2$, let $\psi \not\equiv 0$ be a spinor such that $\nabla^{(f)} \psi = 0$, where f is a real-valued function. Then f is a constant and ψ is a Killing spinor. The manifold is compact if this spinor is non trivial.* ∎

6. The case where n is $n \geqslant 3$

a) If n is $\geqslant 3$ we have $au = C$ (see (4.7)) and $a \nabla_\alpha u + u \nabla_\alpha a = 0$. According to (4.4), the relation (5.3) can be written as:

$$u\nabla_\beta a + n\, a\nabla_\beta u + (i/2)\nabla_\alpha b\, \tilde{\psi}(\gamma^\alpha \gamma_\beta - \gamma_\beta \gamma^\alpha)\,\psi = 0$$

or

$$(n-1)\,u\,\nabla_\beta a - (i/2)\nabla_\alpha b\tilde{\psi}(\gamma^\alpha \gamma_\beta - \gamma_\beta \gamma^\alpha)\,\psi = 0 \tag{6.1}$$

Suppose $C \neq 0$; if such is the case, a is without zero. Multiplying (6.1) by $\nabla^\beta b$ we have $\nabla^\beta a\, \nabla_\beta b = 0$ and thus $\nabla^\beta u \nabla_\beta b = 0$. It follows from (4.4) that $b\, J^\alpha(\psi)\, \nabla_\alpha b = 0$, that is, according to (4.6):

$$b(\rho - a^2 + b^2) = 0$$

Let K be a domain of W on which $(\rho - a^2 + b^2) \neq 0$; we have $b = 0$ on K and, according to (6.1) $a =$ const on K; f being constant on K, it follows from (4.2) that on K

$$R_{\alpha\beta} = (4(n-1)/n^2)\, a^2\, g_{\alpha\beta} = (R/n)\, g_{\alpha\beta}$$

and that $\rho = a^2$. Therefore $(\rho - a^2 + b^2) = 0$ on K, a contradiction. We see that if $a \not\equiv 0$ we have on W

$$\rho - a^2 + b^2 = 0 \tag{6.2}$$

b) It follows from (6.1) that

$$(n-1)u\, \nabla^\beta\, a\, \nabla_\beta a - (i/2)\nabla_\alpha\, b\, \nabla_\beta a \tilde{\psi}(\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha)\psi = 0$$

But we deduce from (5.4), according to (6.2):

$$u\nabla^\beta b \nabla_\beta b - (i/2)\nabla_\alpha a \nabla_\beta b\, \psi(\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha)\,\psi = 0$$

We obtain by addition

$$(n-1)\nabla^{\beta}a\nabla_{\beta}a + \nabla^{\beta}b\nabla_{\beta}b = 0.$$

It follows that $a = \text{const.}$, $b = \text{const.}$ and ψ is a Killing spinor. We have:

THEOREM 2. *On a spin manifold (W, g) of dimension $n \geqslant 3$, let $\psi \not\equiv 0$ be a spinor such that $\nabla^{(f)}\psi = 0$, where f is a complex-valued function with real part $a \not\equiv 0$. Then f is a constant and ψ is a Killing spinor on (W, g) which is necessarily compact.* ∎

For $n \geqslant 3$, we obtain an extension of the above Hijazi proposition.

7. Parallel forms and Killing spinors

a) Let (W, g) be a spin manifold of dimension $n \geqslant 2$. We denote by S the classical isomorphism between forms and (1,1)-spinors; if β is a k-form, $S\beta$ is given by

$$S\beta = (1/k!)\,\beta_{\lambda_1 \dots \lambda_k}\gamma^{\lambda_1}\dots\gamma^{\lambda_k}$$

A straightforward computation shows that:

$$\gamma^{\alpha}(S\beta)\,\gamma_{\alpha} = (-1)^{k-1}n(1-(2k/n))\,(S\beta) \tag{7.1}$$

Let ψ be a non trivial Killing spinor on (W, g) ($\nabla^{(\nu_1)}\psi = 0$). If β is a parallel k-form ($k \neq 0, n$) on (W, g), $S\beta$ is also parallel and we have

$$\nabla_{\alpha}(S\beta)\,\psi = -(\nu_1/n)\,(S\beta)\,\gamma_{\alpha}\psi$$

It follows that:

$$-\nabla^{\alpha}\nabla_{\alpha}(S\beta)\,\psi = (\nu_1^2/n)\,(S\beta)\,\psi$$

We set $\chi = (S\beta)\,\psi$. We have:

$$\Delta\chi = (\nu_1^2/n + R/4)\,\chi \qquad (\nu_1^2 = \rho)$$

or

$$\Delta\chi = \nu_1^2\chi$$

But we deduce from (7.1) that:

$$P\chi = -(\nu_1/n)\,\gamma^{\alpha}(S\beta)\,\gamma_{\alpha}\psi = \nu\chi$$

where $\nu = (-1)^k\,\nu_1(1 - 2k/n)$. It follows $\Delta\,\chi = \nu^2\chi$. If $(S\beta)\psi \not\equiv 0$, we have $\nu^2 = \nu_1^2$ and thus $(1-2k/n)^2 = 1$, impossible for $k \neq 0, n$. Therefore $(S\beta)\,\psi = 0$.

b) We have then $\beta_{\lambda_1 \dots \lambda_k}\gamma^{\lambda_1}\dots\gamma^{\lambda_k}\psi = 0$. It follows by differentiation that:

$$\beta_{\lambda_1 \ldots \lambda_k} \gamma^{\lambda_1} \ldots \gamma^{\lambda_k} \gamma^\alpha \psi = 0$$

that is:

$$(-1)^k \beta_{\lambda_1 \ldots \lambda_k} \gamma^\alpha \gamma^{\lambda_1} \ldots \gamma^{\lambda_k} \psi - 2k\, \beta_{\lambda_1 \ldots \lambda_{k-1}\alpha} \gamma^{\lambda_1} \ldots \gamma^{\lambda_{k-1}} \psi = 0$$

We have:

$$\beta_{\alpha_1 \lambda_2 \ldots \lambda_k} \gamma^{\lambda_2} \ldots \gamma^{\lambda_k} \psi = 0$$

After new differentiations, we get:

$$\beta_{\alpha_1 \alpha_2 \ldots \alpha_{k-1} \lambda_k} \gamma^{\lambda_k} \psi = 0$$

which implies $\beta = 0$. We have:

PROPOSITION 6. *Let (W, g) be a spin manifold of dimension $n \geqslant 2$ admitting a non trivial Killing spinor. There are no non trivial parallel k-forms ($k \neq 0, n$) on (W, g). In particular such a manifold is necessarily irreducible and non-Kählerian.* ■

3. THE HIJAZI INEQUALITY AND ITS UNIVERSALITY

8. Conformal change of metric

In this section, *we suppose that W is compact and that dim W is $\geqslant 3$* [13, 14].

a) Let us come back to the situation and to the notations of paragraph 3. It is convenient to write the conformal factor under the form $\exp(2c) = h^{4/n-2}$, where h is > 0. The scalar curvature $\bar{R}$ of $(W, \bar{g})$ is given by:

$$\bar{R} \exp(2c) = (4(n-1)/(n-2))\, h^{-1} \Delta h + R \tag{8.1}$$

where Δ is the positive Laplacian. We are led to introduce the Yamabe operator

$$Lh = (4(n-1)/(n-2))\, h^{-1} \Delta h + Rh \tag{8.2}$$

It is known that if $\mu_1(g)$ is respectively < 0, null or > 0, there is on W a metric conformal to g with a scalar curvature $R < 0$, null or > 0. If $R =$ const., we have $\mu_1(g) = R$.

Moreover it follows from the theorem of Yamabe-Aubin-Schoen [15] that for $\mu_1(g)$ respectively < 0, null or > 0, there exists on W a metric conformal to g wih a *constant* scalar curvature < 0, null or > 0.

b) Let $\mathscr{H}$ be a space of the *harmonic spinors* of (W, g) (that is such that

$P\psi = 0$ or $\Delta\psi = 0$). It is known (as it is clear on (3.3)) that the complex dimension of $\mathscr{H}$ is a conformal invariant of (W, g), that on a manifold with a scalar curvature R everywhere > 0, we have dim $\mathscr{H} = 0$ and that if $R = 0$, every harmonic spinor is parallel [6].

Let $\mathscr{K}$ be the space of the *twistor-spinors* of (W, g). It follows from (2.7) that on a manifold with a scalar curvature R everywhere < 0, we have dim $\mathscr{K} = 0$ and that if $R = 0$, every twistor-spinor is harmonic and thus is parallel.

We deduce from the conformal invariance of the dimensions of $\mathscr{H}$ and $\mathscr{K}$.

THEOREM 3. *Let (W, g) be a compact spin manifold of dimension $n \geqslant 3$.*
If $\mu_1(g) > 0$ we have dim $\mathscr{H} = 0$
If $\mu_1(g) < 0$ we have dim $\mathscr{K} = 0$
If $\mu_1(g) = 0$ we have dim $\mathscr{K} =$ dim $\mathscr{H}$. ∎

9. Generalization of Hijazi Inequality

a) According to (3.3), we have the following proposition:

PROPOSITION 7. *Let $\psi \not\equiv 0$ be a spinor of (W, g) such that $(P\psi, P\psi) = \lambda^2(\psi, \psi)$* ($\lambda$ = const >0).

For any real-valued function c, the spinor $\bar{\varphi}$ of $(W, \bar{g} = \exp(2c)g)$ corresponding to $\varphi = \exp((-(n-1)/2)c)\psi$ satisfies $(\bar{P}\bar{\varphi}, \bar{P}\bar{\varphi}) = r^2(\bar{\varphi}, \bar{\varphi})$ where $r = \lambda \exp(-c) > 0$. Conversely let $\bar{\varphi}$ be a spinor of $(\bar{W}, \bar{g})$ such that $(\bar{P}\bar{\varphi}, \bar{P}\bar{\varphi}) = r^2(\bar{\varphi}, \bar{\varphi})$, where r is >0; for c such that $r = \lambda \exp(-c)$ the spinor $\psi = \exp((n-1)/2)c)\varphi$ of (W, g) satisfies $(P\psi, P\psi) = \lambda^2(\psi, \psi)$. ∎

b) If $\mu_1(g)$ is the first eigenvalue of the operator L given by (8.2), there is a factor $\exp(2c_1)$ such that $\mu_1(g) = \bar{R}_1 \exp(2c_1)$, where $\bar{R}_1$ is the scalar curvature of $\bar{g} = \exp(2c_1)g$. Let ψ be an arbitrary spinor on (W, g), $\bar{\varphi}$ the conformal spinor of $(W, \bar{g})$ associated with ψ ($\varphi = \exp((-(n-1)/2)c_1)\psi$).

Apply (2.5) to $(W, \bar{g})$ and to $\bar{\varphi}$. We obtain by integration on $(W, \bar{g})$:

$$\int_W [(\bar{R}_1/4)\bar{u} - ((n-1)/n)(\bar{P}\bar{\varphi}, \bar{P}\bar{\varphi})]\bar{\eta} + \int_W (\bar{D}\bar{\varphi}, \bar{D}\bar{\varphi})\bar{\eta} = 0 \tag{9.1}$$

where $\bar{\eta}$ is the volume element of $(W, \bar{g})$ and $\bar{u}$ the square of $\bar{\varphi}$. The first integral is $\leqslant 0$ and the equality corresponds to the case where $\bar{D}\bar{\varphi} = 0$. *We now have*

$$\bar{\eta} = \exp(n\, c_1)\,\eta \qquad \bar{u} = \exp(-(n-1)c_1)u$$

It follows from (9.1) that:

$$\int_W [(\mu_1(g)/4)u - ((n-1)/n)(P\psi, P\psi)] \exp(-c_1)\,\eta \leqslant 0 \tag{9.2}$$

We have:

THEOREM 4. *Let (W, g) be a compact spin manifold of dimension $n \geqslant 3$, $\mu_1(g)$ the first eigenvalue of the Yamabe operator. For any spinor field ψ, there is a domain of W on which:*

$$(P\psi, P\psi) \geqslant (n/4(n-1))\,\mu_1(g), \quad (\psi, \psi) \tag{9.3}$$

If $\psi \not\equiv 0$ is such that $(P\psi, P\psi) = \lambda^2(\psi, \psi)$ ($\lambda = \text{const} > 0$), we have the Hijazi inequality:

$$\lambda^2 \geqslant (n/4(n-1))\,\mu_1(g) \tag{9.4}$$

This theorem is of interest only if $\mu_1(g)$ is *positive*. The inequality (9.4) has been obtained by Hijazi for the particular case of the eigenvalues of the Dirac operator, by a different method [8].

10. The limiting case

Suppose $\mu_1(g) > 0$ and set $\lambda_1^2 = (n/4(n-1))\,\mu_1(g)$ (with $\lambda_1 > 0$).

a) Suppose that we have a spinor $\psi \not\equiv 0$ such that $(P\psi, P\psi) = \lambda_1^2(\psi, \psi)$. We have $(\bar{P}\bar{\varphi}, \bar{P}\bar{\varphi}) = r_1^2(\bar{\varphi}, \bar{\varphi})$ where $r_1 = \lambda_1 \exp(-c_1)$ satisfies

$$(\bar{R}_1/4) - ((n-1)/n)\,r_1^2 = 0 \tag{10.1}$$

It follows from (9.1) that $\bar{D}\bar{\varphi} = 0$. We have $\bar{u} = \text{const.}$ according to (2.10) and according to (2.7)

$$\bar{\Delta}\bar{\varphi} = r_1^2\bar{\varphi} \tag{10.2}$$

Introduce on $(W, \bar{g})$ the spinor $\bar{\chi} = \overline{P\varphi}$ such that $(\bar{\chi}, \bar{\chi}) = r_1^2\bar{u}$. It follows from (10.2) that $(\bar{P}\bar{\chi}, \bar{P}\bar{\chi}) = r_1^2(\bar{\chi}, \bar{\chi})$. We can apply (2.5) to $(W, \bar{g})$ and to $\bar{\chi}$. We obtain according to (10.1).

$$(1/2)\,\bar{\Delta}(r_1^2\bar{u}) + (1/2)\,\bar{\delta}\,\bar{T}(\bar{\chi}) = -(\bar{D}\bar{\chi}, \bar{D}\bar{\chi}) \tag{10.3}$$

and we have by integration $\bar{D}\bar{\chi} = 0$. It follows from (10.1) and (2.10) that $r_1^2\bar{u} =$ $=$ const. *and thus that* $r_1 = $ *const.* $c_1 = $ *const.* By the homothety corresponding to c_1, we deduce from $\bar{u} = \text{const.}$ that $u = \text{const.}$ and from $\bar{D}\bar{\varphi} = 0$ that $D\psi = 0$ according to paragraph 3. It follows from (2.8) and (2.10):

$$\Delta\psi - \lambda_1^2\psi = (P - \epsilon\lambda_1)(P + \epsilon\lambda_1)\,\psi = 0 \tag{10.4}$$

where $\epsilon = \pm 1$. We set

(10.5) $$\alpha_\epsilon = (1/\epsilon \lambda_1) P \psi + \psi$$

where α_{+1} and α_{-1} satisfy

(10.6) $$2\psi = \alpha_{+1} + \alpha_{-1}$$

According to (10.4), we have immediately:

(10.7) $$P \alpha_\epsilon = \epsilon \lambda_1 \alpha_\epsilon$$

b) Let us consider the spinor $\alpha \not\equiv 0$ that $P\alpha = \nu_1 \alpha$, where $\nu_1^2 = (n/4(n-1))$ $\mu_1(g) \neq 0$. It follows from (a) that $D\alpha = 0$ that is

$$\nabla\alpha + (\nu_1/n)\gamma\alpha = \nabla^{(\nu_1)}\alpha = 0$$

LEMMA (Hijazi). *On a compact spin manifold of dimension $n \geqslant 3$, every spinor $\alpha \not\equiv 0$ satisfying $P\alpha = \nu_1\alpha$, where $\nu_1^2 = (n/4(n-1))\,\mu_1(g) \neq 0$, is a non trivial Killing spinor* [8]. ■

According to (10.6), we see that, with the notations of (a), either α_{+1} or α_{-1} is $\not\equiv 0$ and thus is a Killing spinor. It follows for the limiting case of the inequality (9.4).

THEOREM 5. *Let (W, g) be a compact spin manifold of dimension $n \geqslant 3$. If $\psi \not\equiv 0$ is a spinor field on (W, g) such that $(P\psi, P\psi) = \lambda_1^2(\psi, \psi)$, where $\lambda_1^2 = (n/4(n-1))\,\mu_1(g) \neq 0$, (W, g) admits a Killing spinor of the form $(1/\epsilon\lambda_1)$ $P\psi + \psi$ $(\epsilon = \pm 1)$ corresponding to the eigenvalue $\nu_1 = \epsilon\lambda_1$. In particular (W, g) is an Einstein space.* ■

11. Twistor-spinors and Killing spinors

Analyse the space $\mathscr{K}$ of the twistor-spinors by means of the theorem of Yamabe-Shoen [15]. If dim $\mathscr{K} \neq 0$, by a conformal change of the metric and of the twistor spinors, we can suppose $R = \text{const.} \geqslant 0$. If $R = 0$, we have seen that there is coincidence between $\mathscr{K}$ and the space of the Killing spinors (all parallel in this case).

Suppose that $R = \text{const.} > 0$. We have $R = \mu_1(g)$ and $\rho = \lambda_1^2$ (with $\lambda_1^2 = (n/4(n-1))R;\ \lambda_1 > 0$).

If $\psi \not\equiv 0$ belongs to $\mathscr{K}$, we have (10.4) and, with a change of notations, $2\psi = \alpha + \beta$, where

(11.1) $$\alpha = (1/\lambda_1) P\psi + \psi \qquad \beta = -(1/\lambda_1) P\psi + \psi$$

If α, β are $\neq 0$, α and β are Killing spinors. If α or β is null, ψ is a Killing spinor.

Let K_{ν_1} be the space of the spinors ψ satisfying $P\psi = \nu_1 \psi$ (where $\nu_1^2 = n/4(n-1)\mu_1(g)$). If ν_1 and $-\nu_1$ are eigenvalues of P, we have $\mathcal{K} = K_{\nu_1} \oplus K_{-\nu_1}$; it is the case, in particular, if n is even. If ν_1 is eigenvalue and no $-\nu_1$, we have $\mathcal{K} = K_{\nu_1}$ and $K_{-\nu_1} = \{0\}$.

We see by a conformal change of metric that we have:

THEOREM 6. *Let (W, g) be a compact spin manifold of dimension $n \geqslant 3$ such that the space $\mathcal{K}$ of the twistor-spinors of (W, g) is not reduced to 0. There exists on W a metric $\bar{g} = \exp(2c)g$ such that $(W, \bar{g})$ admits non vanishing Killing spinors and thus is an Einstein space. If $\mu_1(g) > 0$, $(W, \bar{g})$ is irreducible and non Kählerian.*

The dimension of $\mathcal{K}$ is the dimension of the space $K_{\nu_1} \oplus K_{-\nu_1}$, where K_{ν_1} (resp. $K_{-\nu_1}$ if it is $\neq 0$) is the space of the Killing spinors of $(W, \bar{g})$ corresponding to ν_1 (resp. $-\nu_1$). ■

12. Zeros of a twistor-spinor

Suppose (W, g) *such that* $R = const. > 0$. Let $\psi \not\equiv 0$ be a twistor-spinor of (W, g). If ν_1 and $-\nu_1$ (with $\nu_1^2 = \rho$) are eigenvalues of the Dirac operator P, it follows from paragraph 11 that $\psi = \alpha + \beta$, where α (resp. β) is a Killing spinor corresponding to the eigenvalue ν_1 (resp. $-\nu_1$).

Set $f = (\alpha, \beta)$ and study Δf. We have:

$$\nabla_\lambda f = \nabla_\lambda \tilde{\beta} \cdot \alpha + \tilde{\beta} \nabla_\lambda \alpha$$

where

$$\nabla_\lambda \alpha = -(\nu_1/n)\gamma_\lambda \alpha \qquad \nabla_\lambda \tilde{\beta} = -(\nu_1/n)\tilde{\beta}\gamma_\lambda$$

It follows:

$$\Delta f = -\nabla^\lambda \nabla_\lambda \tilde{\beta} \cdot \alpha - \tilde{\beta} \nabla^\lambda \nabla_\lambda \alpha - 2\nabla_\lambda \tilde{\beta} \nabla^\lambda \alpha$$

where:

$$2\nabla_\lambda \tilde{\beta} \nabla^\lambda \alpha = -(2\nu_1^2/n) f$$

Moreover

$$-\nabla^\lambda \nabla_\lambda \alpha = (\nu_1/n) P\alpha = (\nu_1^2/n)\alpha \qquad -\nabla^\lambda \nabla_\lambda \tilde{\beta} = (\nu_1^2/n)\tilde{\beta}$$

We obtain:

$$\Delta f = (4\nu_1^2/n) f \tag{12.1}$$

that is

(12.2) $\Delta f = (R/(n-1)) f$

It is known (theorem of Obata-Lichnerowicz [16, 18]) that, if the complex-valued function f, solution of (12.2) is not identically zero, (W, g) is necessarily isometric with the sphere $(S^n$, Can) endowed with its canonical metric. If such is not the case, we have $f = (\alpha, \beta) \equiv 0$. Therefore

$$(\psi, \psi) = (\alpha, \alpha) + (\beta, \beta)$$

and since $(\alpha, \alpha) = \text{const.}$, $(\beta, \beta) = \text{const.}$, we have

$$(\psi, \psi) = \text{const.}$$

If ψ admits a zero on W, we have $\psi \equiv 0$.

We obtain by means of a conformal change of metric and of spinor

THEOREM 7. *Let (W, g) be a compact spin manifold of dimension $n \geqslant 3$ which is not conformally isometric to the sphere* $(S^n$, Can). *Every twistor-spinor $\psi \equiv 0$ of (W, g) is without zero on W.*

It is clear that, for $(S^n$, Can) the conclusion of the theorem does not hold. We note that, for $n = 4$, if (W, g) admits a twistor-spinor $\not\equiv 0$, (W, g) is conformally isometric with $(S^4$, Can) (see Hijazi [17]). For $n = 5$, there exist, according to S. Sutanke nonhomogeneous manifolds S^5/Γ (Γ discrete) admitting non trivial twistor-spinors. For $n = 6$, T. Friedrich and R. Grunewald have showed that $P_3(C)$ (and F(1, 2)) endowed with a suitable metric admits twistor-spinors $\not\equiv 0$ and thus without zero.

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